

# Prime Decomposition Chains

## The 'Sum Of Prime Factors' Function Iterates To Form Sequences Of Sums

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### Prime Decompositions and $sopf(n)$

The *prime decomposition* of a number entails its factorization into its constituent primes (also called *prime factorization*). A formal ([Mathworld](#)) definition says:

Given a positive integer  $n \geq 2$ , the prime factorization is written

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$$

where the  $p_i$ s are the prime factors, each of order  $a_i$ . Each factor  $p_i^{a_i}$  is called a primary.

The *sum of prime factors* function  $sopf(n) = g(f(n))$ , where  $f(n) = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$  is followed by  $g(p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}) = a_1 p_1 + a_2 p_2 \cdots + a_k p_k = n'$ . That is,  $g$  "downgrades" the operators in the decomposition. E.g.,  $sopf(18) = 2 + 3 \cdot 2 = 8$ . (See Sloane's [A008472](#).)

Restated,  $g$  in  $g(f(n))$  converts the prime decomposition's exponents to coefficients and its multiplication operators to addition:  $g(p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}) = a_1 p_1 + a_2 p_2 \cdots + a_k p_k$ .

Next,  $n$  and iteration of  $sopf(n)$  defines the *factorization sequence* function  $fs(n)$ . That is,  $sopf(21) = 10$ ,  $sopf(10) = 7$ ,  $sopf(7) = 7$ . When  $sopf(n) = n$ , the procedure terminates, as  $n$  is thus prime or else equal to 4. While  $n \neq 4$  is composite, the process repeats. Hence,  $fs(21) = 21, 10, 7$ . For another example;  $fs(33122169208733534500868101244643841) = 33122169208733534500868101244643841, 131463022833, 74147424, 157$ .

### Some Transforms Based on $fs(n)$

We next let  $n = 1, 2, 3, \dots$  and use  $fs(n)$  to transform this sequence in two ways. One substitutes the length ( $j$ ) of the factorization chain for  $n$  and the other substitutes the number ( $n_j$ ) that terminates it.

Table 1:  $n = 1, 2, 3, \dots$

$n_1 =$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$j =$	1	1	1	1	2	1	3	3	2	1	2	1	4	4	4	1	4	1	4	3	2
$n_j =$	2	3	4	5	5	7	5	5	7	11	7	13	5	5	5	17	5	19	5	7	13

Table 1 gives the  $j$  sequence as 1, 1, 1, 1, 2, 1, 3, 3, 2, 1, 2, 1, 4, 4, 4, 1, 4, 1, 4, 3, 2... (see Sloane's [A002217](#)) and the  $n_j$  sequence as 2, 3, 4, 5, 5, 7, 5, 5, 7, 11, 7, 13, 5, 5, 5, 17... (Sloane's [A029908](#)). As these [OEIS](#) pages show, these particular transforms have been looked at closely. In the patterns below,  $fs(n)$  applies to other common sequences:

Table 2:  $n = 1, 2, 3, 5, 8, \dots$

$n_1 =$	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181
$j =$	1	1	1	3	1	3	2	5	1	5	1	4	5	4	1	4	6
$n_j =$	2	3	5	5	13	7	19	5	89	5	233	7	7	13	1597	7	5

Table 3:  $n = 1, 3, 6, 10, \dots$

$n_1 =$	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190
$j =$	1	2	2	4	3	2	3	2	5	5	5	5	5	5	2	2	4	6
$n_j =$	3	5	7	5	7	11	7	11	5	5	5	5	5	5	23	23	7	5

Note that for the triangular numbers (above), some vague semblance of order seems to be creeping into the  $j$  and  $n_j$  patterns. This shows up in the square number patterns (below) as well.

Table 4:  $n = 1, 4, 9, 16, \dots$

$n_1 =$	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289
$j =$	1	3	3	3	3	5	3	3	5	3	5	6	5	5	5	3
$n_j =$	4	5	7	7	7	5	7	7	5	13	5	5	5	5	5	19

Of the 100,000+ sequences in the [OEIS](#), are there any for which  $fs(n)$  generates patterns that are entirely regular and predictable?

### Sequence Summation Chains

Our next avenue of exploration adds up all of the elements of an  $fs(n)$  expansion and, in turn, applies  $fs(n)$  to the sum. This creates monotonically increasing chains that terminate only at a prime. We'll refer to these chains as summation chains, and represent them (and the function that produces them) as  $s(n)$ . In the examples below, the top line of each table holds the sequence of interest; e.g.,  $s(22) = 22, 35, 54, 65, 102, 137$ .

Table 5: Some Examples of Summation Chains

$n_1 =$	22	35	54	65	102	137	28	39	74	148	189	224	241
$n_2$	13	12	11	18	22		11	16	39	41	16	17	
$n_3$		7		8	13		8	16			8		
$n_4$				6			6	8			6		
$n_5$				5			5	6			5		
$n_6$								5					

  

$n_1 =$	496	570	599	755	951	2043	2276	3159	3198	3257
$n_2$	39	29		156	320	233	573	28	59	
$n_3$	16			20	17		194	11		
$n_4$	8			9			99			
$n_5$	6			6			17			
$n_6$	5			5						

Below are the  $s(n)$  that derive from the first few composite  $n$ :

$s(6) = 6, 11$ ;  $s(8) = 8, 19$ ;  $s(9) = 9, 20, 40, 51, 91, 131$ ;  $s(10) = 10, 17$ ;  $s(12) = 12, 19$ ;  $s(14) = 14, 34, 53$ ;  $s(15) = 15, 34, 53$ ;  $s(16) = 16, 35, 54, 65, 102, 137$ ;  $s(18) = 18, 37$ ;  $s(20) = 20, 40, 51, 91, 131$ ;  $s(21) = 21, 38, 76, 99, 116, 183, 266, 305, 406, 482, 759, 796, 1052, 1458, 1498, 1681, 1806, 1896, 2001, 2091, 2152, 2465, 2556, 2656, 2802, 3376, 3694, 5685, 6155, 7538, 11781, 11822, 12173, 12304, 13128, 13871, 14027, 14136, 14195, 14419$ ;  $s(22) = 22, 35, 54, 65, 102, 137$ ;  $s(24) = 24, 44, 78, 115, 154, 194, 310, 386, 619$ ;  $s(25) = 25, 42, 61$ ;  $s(26) = 26, 60, 79$ ;  $s(27) = 27, 47$ ;  $s(28) = 28, 39, 74, 148, 189, 224, 241$ ;  $s(30) = 30, 47$ ;  $s(32) = 32, 49, 83$ ;  $s(33) = 33, 67$ ;  $s(34) = 34, 53$ ;  $s(35) = 35, 54, 65, 102, 137$ ;  $s(36) = 36, 53$ ;  $s(38) = 38, 76, 99, 116, 183, 266, 305, 406, 482, 759, 796, 1052, 1458, 1498, 1681, 1806, 1896, 2001, 2091, 2152, 2465, 2556, 2656, 2802, 3376, 3694, 5685, 6155, 7538, 11781, 11822, 12173, 12304, 13128, 13871, 14027, 14136, 14195, 14419$ ;  $s(39) = 39, 74, 148, 189, 224, 241$ ;  $s(40) = 40, 51, 91, 131$ ;  $s(42) = 42, 61$ ;  $s(44) = 44, 78, 115, 154, 194, 310, 386, 619$ ;  $s(45) = 45, 56, 69, 129, 217, 293$ .

These numbers may be organized into tables to create new sequences. In table 6 (below),  $t$  = the length of a given  $n$ 's sequence, and  $t_j$  = its last term.

Table 6

$n =$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$t =$	1	1	1	1	2	1	2	6	2	1	2	1	3	3	6	1	2	1	5	40	6
$t_j =$	2	3	4	5	11	7	19	131	17	11	19	13	53	53	137	17	37	19	131	14419	137
$n =$	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40			
$t =$	1	9	3	3	2	7	1	2	1	3	2	2	5	2	1	39	6	4			
$t_j =$	23	619	61	79	47	241	29	47	31	83	67	53	137	53	37	14419	241	131			

It could be interesting to see what happens to the ratio  $n/t$  as  $n$  grows large. We know for prime values of  $n$  that  $t = 1$ , and hence  $n/t$  will approach infinity. But what of composites? For  $n = 21$  and  $t = 40$ ,  $n/t = 0.525\dots$ , only the second instance in the table where  $n/t$  is  $< 1$ . Is there, in theory, a non-zero lower bound on this ratio? If a chain somehow manages to thread its way through the infinite minefield of primes that lies beyond it (i.e., fails to terminate), then  $\lim_{n \rightarrow \infty} \frac{n}{t} = 0$ , but what of finite  $s(n)$ ?

This brings up the question, can it be proved that  $s(n)$  always terminates? How can we know that a chain on the order of, say,  $s(\text{googolplex}^{\text{googolplex}})$  will inevitably encounter one of the ever scarcer primes?

*Problems:*

Prove that  $s(n)$  terminates for all  $n$ .

What are the smallest  $n/t$  and  $n/t_j$  ratios for  $n < 10^7$ .

How does the average size of  $t$  change as  $n$  grows large?